Conflict Resolution with Poisson Arrivals

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1 Introduction

(see [2] for a survey)

How quickly can you *separate* the events in a Poisson point process (assuming ternary feedback- silence, success and collision)? Specifically, let f be an algorithm and $A > 0$. Let $N(f, A)$ be the expected number of queries to separate all the events in $[0, A]$. We define the capacity of f to be

$$C_f = \lim_{A\to\inf} A/N(f, A),$$

assuming it exists for reasonable algorithms. We are interested in the optimal capacity, i.e.

$$C^* = \inf_f C_f.$$

Trivially, $C^* \leq 1$, since you need at least one query to separate one event. The quite-trivial tree algorithm has capacity 0.487. Note that you can slightly increase (less than 0.01) its capacity by some easy trick. Scholars use potential method to prove its upper bounds. Specifically, we know

$$0.587 > C^* > 0.487.$$

But what is C^* exactly? After the novel 0.587-0.487 upper-lower bound, no significant progress was made for 40 years already (1981-2020).

1.1 Entropy as Potential

Assume the optimal algorithm has the average feedback frequency (a, b, c) where $a+b+c = 1$ (thus $C^* = b$). By the Log sum inequality, the entropy of feedback is bounded above by $A(-a \log a - b \log b - c \log c)$.

1.2 Soviet Lemma

Mikhailov and Tsybakov [1] prove a strong lemma about the distribution of the feedback. Let ϕ_0 and ϕ_1 be the probability that it gets silence and success. Let ε be the size of the query set. Then, independent from history, we have

$$\phi_0 \leq e^{-\varepsilon}, \quad \text{and} \quad \phi_1 \leq \frac{\varepsilon e^{-\varepsilon}}{1 - e^{-\varepsilon}}(1 - \phi_0).$$

It is easy to see ϕ_0's upper bound but the bound for ϕ_1 is not apparent. I rephrase the proof here for reference, since the original proof was written in Russian.

Proof. Let B_t be the tth query set and $\theta_t(B_t)$ be the feedback. Let $D_t = \bigcup_{s=1;\theta_s(B_s)\neq 2}^{t} B_s$ be the resolved set after t queries. Let $\varepsilon = \text{mes}(B_t \setminus D_{t-1})$ be the (effective) size of tth query set.

Let $\{C^i\}_{i=1}^N$ be a partition of $\bigcup_{s=1}^t B_s$ such that each B_s can be written as a disjoint union of some subset of $\{C^i\}_{i=1}^N$. Specifically, let $I = \{i_1, \ldots, i_M\}$ and $B_t = \bigcup_{j=1}^M C^{i_j}$.

Let $\bar{I} = [N] \setminus I$. Let $\eta = \{\theta_0(C^i) \mid i \in \bar{I}\}$ be the ternary state of each C^i. There are totally $K = 3^{N-M}$ possibilities of η, which we denote as $\eta_1, \eta_2, \ldots, \eta_K$.

$$\mathbb{P}(\theta_t(B_t) = y \mid \{\theta(t-1)\}) = \sum_{k=1}^{K} \mathbb{P}(\theta_t(B_t) = y \mid \eta = \eta_k, \{\theta(t-1)\}) p_k,$$

where $p_k = \mathbb{P}(\eta = \eta_k \mid \{\theta(t-1)\})$. $p_k = 0$ if η_k is incompatible with $\{\theta(t-1)\}$. It suffices to prove that for each η_k with $p_k > 0$,

$$\mathbb{P}(\theta_t(B_t) = 0 \mid \eta = \eta_k, \{\theta(t-1)\}) \leq e^{-\varepsilon}, \tag{1}$$

$$\mathbb{P}(\theta_t(B_t) = 1 \mid \eta = \eta_k, \{\theta(t-1)\}) \leq \frac{\varepsilon e^{-\varepsilon}}{1 - e^{-\varepsilon}} (1 - \mathbb{P}(\theta_t(B_t) = 0 \mid \eta = \eta_k, \{\theta(t-1)\})). \tag{2}$$

We partition the set of queries $\{1, 2, \ldots, t-1\}$ into 5 disjoint sets W_0, W_1, W_2, W_3 and W_4 where

$$\begin{aligned}
W_y &= \{s \mid \theta_s(B_s) = y\}, \quad y = 0, 1, \\
W_2 &= \{s \mid \theta_s(B_s) = 2, \sum_{i \in \bar{I}, C^i \subset B_s} \theta_0(C_i) \geq 2\} \\
W_3 &= \{s \mid \theta_s(B_s) = 2, \sum_{i \in \bar{I}, C^i \subset B_s} \theta_0(C_i) = 1\} \\
W_4 &= \{s \mid \theta_s(B_s) = 2, \sum_{i \in \bar{I}, C^i \subset B_s} \theta_0(C_i) = 0\}.
\end{aligned}$$

Note that given η_k, the feedback $\{\theta_s(B_s) \mid s \in W_0 \cup W_1 \cup W_2\}$ does not affect the current query and we only need to look at W_3 and W_4.

1. If $W_4 \neq \emptyset$, which means at least two points lie in B_t, i.e.

   $$\mathbb{P}(\theta_t(B_t) = y \mid \eta = \eta_k, \{\theta(t-1)\}) = 0, \quad y = 0, 1,$$

   which satisfies inequalities (1) and (2).

2. If $W_3 \cup W_4 = \emptyset$, which means it is a query of a fresh Poisson random variable with mean ε. Inequalities (1) and (2) are trivially satisfied.

3. Finally, if $W_4 = \emptyset$ but $W_3 \neq \emptyset$, we first have $\mathbb{P}(\theta_t(B_t) = 0 \mid \eta = \eta_k, \{\theta(t-1)\}) = 0$, since there exists at least one point in B_t (inequality (1) satisfied). Let $R = B_t \cap \left(\bigcap_{s \in W_3} B_s\right)$ and $\delta = |R|$. In this case, we have

   $$\mathbb{P}(\theta_t(B_t) = 1, \eta = \eta_k, \{\theta(t-1)\}) = \mathbb{P}(\eta = \eta_k) \delta e^{-\delta} e^{-(\varepsilon - \delta)},$$

   and

   $$\mathbb{P}(\eta = \eta_k, \{\theta(t-1)\}) = \mathbb{P}(\eta = \eta_k)(1 - e^{-\delta}).$$

   Combining these two we have

   $$\mathbb{P}(\theta_t(B_t) = 1 \mid \eta = \eta_k, \{\theta(t-1)\}) \leq \frac{\delta e^{-\varepsilon}}{1 - e^{-\delta}} \leq \frac{\varepsilon e^{-\varepsilon}}{1 - e^{-\varepsilon}}.$$

   Thus inequality (2) also satisfied (note that here $\phi_0 = 0$).

□

References

[1] Viktor Alexandrovich Mikhailov and Boris Solomonovich Tsybakov. Upper bound for the capacity of a random multiple access system. *Problemy Peredachi Informatsii*, 17(1):90–95, 1981.

[2] Mart L Molle and George C Polyzos. Conflict resolution algorithms and their performance analysis. *University of Toronto, CS93-300, Tech. Rep*, 1993.