Equivalence of Linearizability Definitions

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For any $n \in \mathbb{N}$, we define $[n]$ to be the set $\{1, 2, \ldots, n\}$. Of course, $[0] = \emptyset$.

Definition (transactions)**.** Let the set of n transactions be the set of intervals $\{[a_k, b_k]\}_{k\in[n]}$ where $a_k, b_k \in \mathbb{R}^+$ and $a_k \le b_k$ for all $k \in [n]$. For the transaction k, it starts at time a_k and ends at time b_k.

Definition (order linearization)**.** An order linearization of a set of transactions $\{[a_k, b_k]\}_{k\in[n]}$ is an injective order function $f : [n] \to [n]$ such that for all $k, m \in [n]$ if $b_k < a_m$, then $f(k) < f(m)$.

Definition (point linearization)**.** A point linearization of a set of transactions $\{[a_k, b_k]\}_{k\in[n]}$ is a choice function $g : [n] \to \mathbb{R}^+$ such that for all $k \in [n]$, $g(k) \in [a_k, b_k]$.

Of course, not all linearization are valid. However, here we don't need to care about the validity of the linearizations. We just want to prove that for all point linearization there exists an order linearization such that $\forall k, m \in [n]$, $g(k) < g(m) \implies f(k) < f(m)$ and conversely for all order linearization there exists a point linearization such that $\forall k, m \in [n]$, $f(k) < f(m) \implies g(k) \le g(m)$.

Lemma 1. *Given a set of transactions $\{[a_k, b_k]\}_{k\in[n]}$ and a point linearization $g : [n] \to \mathbb{R}^+$, there exists an order linearization $f : [n] \to [n]$ such that $\forall k, m \in [n]$, $g(k) < g(m) \implies f(k) < f(m)$.*

Proof. We just get the permutation $f : [n] \to [n]$ by sorting $g(i)$s so that $\forall k, m \in [n]$, $g(k) < g(m) \implies f(k) < f(m)$ (with arbitrary tie-breaking). Permutation is naturally injective. Now we are left to prove that f is an order linearization. Indeed, for any $k, m \in [n]$, if $b_k < a_m$, then since $g(k) \in [a_k, b_k]$ and $g(m) \in [a_m, b_m]$, we have $g(k) < g(m)$ and thus $f(k) < f(m)$. □

Lemma 2. *Given a set of transactions $\{[a_k, b_k]\}_{k\in[n]}$ and an order linearization $f : [n] \to [n]$, there exists a point linearization $g : [n] \to \mathbb{R}^+$ such that $\forall k, m \in [n]$, $f(k) < f(m) \implies g(k) \le g(m)$.*

Proof. We prove this by induction on n. When $n = 1$, let $g(1) = a_1$ and the statement is trivially true.

Assume the statement is true for $n = t$ with $t \ge 1$ and now we consider a set of $(t + 1)$ transactions $\{[a_k, b_k]\}_{k\in[t+1]}$ and an order function $f : [t + 1] \to [t + 1]$. Since f is bijective, we define $\tau = f^{-1}(1)$. For any $k \in [t + 1]$, we must have $a_\tau \le b_k$. If not, we have $b_k < a_\tau$ which implies $f(k) < f(\tau)$. However $f(\tau) = 1$ is the smallest element in the range. A contradiction.

Consider the set of transactions $T' = \{[\max(a_k, a_\tau), b_k]\}_{k\in[t+1]\setminus\{\tau\}}$ and the restricted order function $f' = f\,|_{[t+1]\setminus\{\tau\}}$. Note that each interval is nonempty since $a_\tau \le b_k$ for all k. We claim that f' is an order linearization of T'. Indeed, for any $k, m \in [t + 1] \setminus \{\tau\}$, if $b_k < \max(a_m, a_\tau)$, since $b_k \ge a_\tau$, then we must have $b_k < a_m$, which implies $f(k) < f(m)$, which implies $f'(k) < f'(m)$.

By the inductive hypothesis there exists a corresponding choice function $g' : [t + 1] \setminus \{\tau\} \to \mathbb{R}^+$ of f' on T'. Now we claim that the function $g : [t + 1] \to \mathbb{R}^+$

$$g(k) = \begin{cases} g'(k), & k \ne \tau \\ a_k, & k = \tau \end{cases}$$

is a choice function such that $\forall k, m \in [t + 1]$, $f(k) < f(m) \implies g(k) \le g(m)$. Indeed, if $k, m \ne \tau$, then it is true by the construction. Now let $k \ne \tau$, then $f(\tau) < f(k)$. We are left to prove that $g(\tau) \le g(k)$, which is trivial since $g(k) = g'(k) \in [\max(a_k, a_\tau), b_k]$ by definition and $g(\tau) = a_k$. □

Having these two lemmas makes the two definitions of linearizability equivalent. Since no matter which kind of linearization you have, you can construct the other kind of linearization with the same total order, using the ways in the proofs- these are constructive proofs.